

Homework 12 Solutions

Math 121A, Fall 2025.

Due: Wednesday, Nov 19, 11:59 PM

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. (*Fourier Transform Diagonalizes Translations*)

(a)

$$\widehat{\tau_h f} = \widehat{f(x-h)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-h) e^{-ikx} dx = \frac{e^{-ikh}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-h) e^{-ik(x-h)} dx = e^{-ikh} \hat{f}.$$

Now for the other equality.

$$\widehat{e^{ihx} f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ihx} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k-h)x} dx = \hat{f}(k-h) = \tau_h \hat{f}.$$

(b) Let's write it: given a function f we have

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x) - f(x-h)}{h} = \lim_{h \rightarrow 0} \frac{f(x) - (\tau_h f)(x)}{h} = \left(\lim_{h \rightarrow 0} \frac{\mathbb{1} - \tau_h}{h} \right) f(x),$$

so we have

$$\frac{d}{dx} = \left(\lim_{h \rightarrow 0} \frac{\mathbb{1} - \tau_h}{h} \right).$$

(c) Using the limit property¹ and linearity of the Fourier Transform,

$$\widehat{\frac{d}{dx} f}(k) = \left(\lim_{h \rightarrow 0} \frac{\mathbb{1} - \tau_h}{h} f \right)(k) = \left(\lim_{h \rightarrow 0} \frac{\mathbb{1} - e^{-ikh}}{h} \hat{f} \right)(k) = ik \hat{f}(k)$$

Problem 2. (*Frequency modulation*)

$g(x) = f(ax)$, let's compute.

$$\begin{aligned} \hat{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-ikx} dx \\ &= \frac{1}{|a| \sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tilde{x}) e^{-ik\tilde{x}/a} d\tilde{x} \\ &= \frac{1}{|a|} \hat{f}\left(\frac{k}{a}\right). \end{aligned}$$

where we use a change of variables $ax = \tilde{x}$ in the second line.

¹A nice way to say this is that the Fourier transform is continuous.

Problem 3. (*Fourier Transform of the Triangle Function*)

$$T(x) = \begin{cases} 2a - |x|, & |x| < 2a, \\ 0, & |x| > 2a, \end{cases}$$

Let's compute.

$$\begin{aligned} \sqrt{2\pi}\hat{T}(k) &= \int_{-2a}^{2a} (2a - |x|)e^{-ikx} dx \\ &= \int_{-2a}^0 (2a + x)e^{-ikx} dx + \int_0^{2a} (2a - x)e^{-ikx} dx \\ &= \frac{2iak - e^{2iak} + 1}{k^2} - \frac{2iak + e^{-2iak} - 1}{k^2} \\ &= \frac{2 - 2\cos(2ak)}{k^2} \\ &= \frac{4\sin^2(ak)}{k^2} \\ &= \frac{4a^2\sin^2(ak)}{(ak)^2} \\ &= 4a^2\text{sinc}^2(ak). \end{aligned}$$

where we have used the trig identity $1 - \cos(2\kappa) = 2\sin^2(\kappa)$.

Problem 4. (*Poisson's Summation Formula*)

(a) We first verify 2π -periodicity.

$$h(x + 2\pi) = \sum_{k=-\infty}^{\infty} f(x + 2(k+1)\pi) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi) = h(x).$$

(this would not have worked if we were not summing over all the integers). So, h is 2π -periodic and (assumed to be) square-integrable, so $h \in L^2([-\pi, \pi])$. The Fourier coefficients are then

$$\begin{aligned} \hat{h}(k) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x)e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} f(x + 2n\pi)e^{-ikx} dx \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{(2n-1)\pi}^{(2n+1)\pi} f(x)e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \hat{f}(k). \end{aligned}$$

(b) Evaluating at $x = 0$,

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = h(0) = \sum_{k=-\infty}^{\infty} \hat{h}(k)e^{ik(0)} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{f}(k).$$

Problem 5. (*A problem from optics*)

Follow the hint, letting

$$f(x) = \begin{cases} 2a - |x| & |x| < 2a, \\ 0 & |x| > 2a \end{cases},$$

which is just the triangle function from earlier and so has Fourier transform

$$\hat{f}(k) = \sqrt{2\pi} \frac{2}{\pi} \frac{\sin^2(ak)}{k^2}.$$

Let $a = \theta$. Then using Poisson summation formula from the previous problem (and evaluating $h(0)$),

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = 2\theta = \sum_{n=-\infty}^{\infty} \frac{\sin^2 n\theta}{n^2} \cdot \frac{2}{\pi}.$$

So in particular

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 n\theta}{n^2} = \pi\theta.$$

Problem 6. (*An exponentially decaying interaction*)

Off we go.

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx \\ &= \frac{2}{\sqrt{2\pi}} \operatorname{Re} \int_0^{\infty} e^{-x-ikx} dx \\ &= \frac{2}{\sqrt{2\pi}} \operatorname{Re} \frac{1}{1+ik} \\ &= \frac{2}{\sqrt{2\pi}} \frac{1}{1+k^2}. \end{aligned}$$

(A small contour argument is necessary for the third equality). Then, the inverse Fourier transform is given by

$$f(x) = \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+k^2} e^{ikx} dk = e^{-|x|}.$$

Taking real parts of both sides and using that $\cos, |x|$ are even,

$$\int_0^{\infty} \frac{\cos kx}{k^2 + 1} dk = \frac{\pi}{2} e^{-|x|}.$$

Problem 7. We recall that

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Observe that this function is closely related to the box function from class: if

$$B(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases},$$

then $f(x) = B\left(\frac{x}{2} - \frac{1}{2}\right) = \tau_{1/2}B\left(\frac{x}{2}\right)$. Thus, using problems 1 and 2 combined with $\hat{B}(k) = \sin(k)/k$,

$$\hat{f}(k) = \tau_{1/2}\widehat{B(x/2)} = e^{-ik/2}2\hat{B}(2k) = e^{-ik/2}2\frac{\sin 2k}{2k} = e^{-ik/2}\frac{\sin 2k}{k}.$$

Problem 8. Compute the convolution of the “rectangular pulse” from class

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

with itself in two ways:

(a) Let’s directly compute $f * f$.

$$(f * f)(x) = \int_{-\infty}^{\infty} f(x-y)f(y) dy.$$

Using the “flip, slide, integrate” thing,

$$(f * f)(x) = \begin{cases} \int_{-1}^{x+1} dy, & 0 \leq x \leq 2 \\ \int_{x-1}^1 dy, & -2 \leq x \leq 0 \end{cases},$$

which simplifies to

$$(f * f)(x) = \begin{cases} 2 - |x| & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$

Yup, it’s the triangle function. Let’s call it T .

(b) The Fourier transform of the box was computed in class:

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \frac{\sin(k)}{k}.$$

Convolution turns into multiplication in the Fourier domain:

$$\widehat{f * f}(k) = \sqrt{2\pi} \hat{f}^2 = \sqrt{2\pi} \frac{2 \sin^2(k)}{\pi k^2},$$

which we can inverse Fourier transform using Problem 4:

$$f * f(x) = T(x),$$

indeed retrieving the promised triangle function.

Problem 9. (*Fourier transform of a Gaussian is a Gaussian*) As the hint suggests, we manipulate the integral by completing the square:

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-ikx} dx \\ &= e^{-k^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x+ik)^2/2} dx \\ &= f(k)I(k), \end{aligned}$$

where $I(k) = \int_{-\infty}^{\infty} e^{-(x+ik)^2/2} dx$. We know from the Gaussian integral that $I(0) = \sqrt{2\pi}$, so our conclusion will follow if we can show that $I(k) = I(0)$ for every real k . To see this, we consider a rectangle with vertices $R, R+ik, -R+ik, -R$, oriented positively. Label the bottom side I, the right side II, the top III, and the left side IV. Since the gaussian is entire, by Cauchy's theorem we know that

$$0 = \int_{\text{I+II+III+IV}} e^{-z^2/2} dz,$$

and that

$$\lim_{R \rightarrow \infty} \int_{\text{I}} e^{-z^2/2} dz = I(0), \quad \lim_{R \rightarrow \infty} \int_{\text{III}} e^{-z^2/2} dz = -I(k).$$

So, it suffices to show that the side wall integrals tend to zero as $R \rightarrow \infty$. We show the case for the right side II and $k > 0$, but the other cases are very similar.

$$\begin{aligned} \left| \int_{\text{II}} e^{-z^2/2} dz \right| &= \left| \int_0^k e^{-(R+iy)^2/2} i dy \right| \\ &= \left| \int_0^k e^{-R^2/2} e^{-iRy} e^{y^2/2} i dy \right| \\ &\leq \int_0^k e^{-R^2/2} e^{k^2/2} dy \\ &= k e^{-R^2/2} e^{k^2/2} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$